# 2021 AMC 12A (Fall Contest) Problems

**Problem 1**

What is the value of $\frac{(2112-2021)^2}{169}$?

$\textbf{(A)}\ 7 \qquad \textbf{(B)}\ 21 \qquad \textbf{(C)}\ 49 \qquad \textbf{(D)}\ 64 \qquad \textbf{(E)}\ 91$

**Problem 2**

Menkara has a $4 \times 6$ index card. If she shortens the length of one side of this card by $1$ inch, the card would have area $18$ square inches. What would the area of the card be in square inches if instead she shortens the length of the other side by $1$ inch?

$\textbf{(A)}\ 16 \qquad \textbf{(B)}\ 17 \qquad \textbf{(C)}\ 18 \qquad \textbf{(D)}\ 19 \qquad \textbf{(E)}\ 20$

**Problem 3**

Mr. Lopez has a choice of two routes to get to work. Route A is $6$ miles long, and his average speed along this route is $30$ miles per hour. Route B is $5$ miles long, and his average speed along this route is $40$ miles per hour, except for a $\frac{1}{2}$-mile stretch in a school zone where his average speed is $20$ miles per hour. By how many minutes is Route B quicker than Route A?

$\textbf{(A)}\ 2\frac{3}{4} \qquad \textbf{(B)}\ 3\frac{3}{4} \qquad \textbf{(C)}\ 4\frac{1}{2} \qquad \textbf{(D)}\ 5\frac{1}{2} \qquad \textbf{(E)}\ 6\frac{3}{4}$

**Problem 4**

The six-digit number $\underline{2}\,\underline{0}\,\underline{2}\,\underline{1}\,\underline{0}\,\underline{A}$ is prime for only one digit $A$. What is $A$?

$(\textbf{A})\ 1 \qquad (\textbf{B})\ 3 \qquad (\textbf{C})\ 5 \qquad (\textbf{D})\ 7 \qquad (\textbf{E})\ 9$

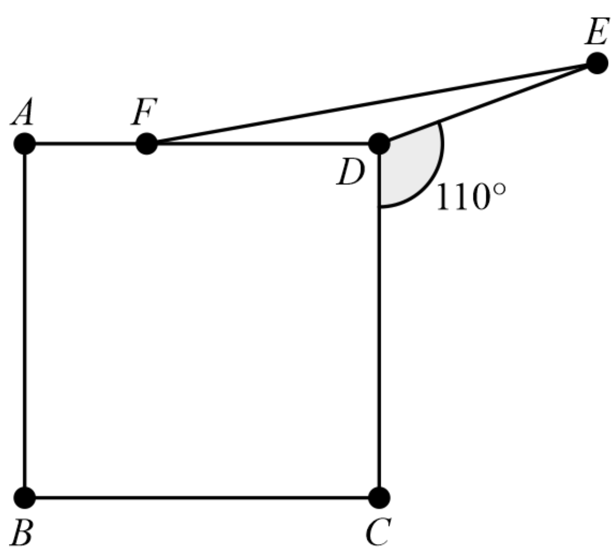
**Problem 5**

Elmer the emu takes $44$ equal strides to walk between consecutive telephone poles on a rural road. Oscar the ostrich can cover the same distance in $12$ equal leaps. The telephone poles are evenly spaced, and the $41$st pole along this road is exactly one mile ( $5280$ feet) from the first pole. How much longer, in feet, is Oscar's leap than Elmer's stride?

$(\textbf{A})\ 6 \qquad (\textbf{B})\ 8 \qquad (\textbf{C})\ 10 \qquad (\textbf{D})\ 11 \qquad (\textbf{E})\ 15$

**Problem 6**

As shown in the figure below, point $E$ lies in the opposite half-plane determined by line $CD$ from point $A$ so that $\angle CDE = 110^\circ$ . Point $F$ lies on $\overline{AD}$ so that $DE = DF$, and $ABCD$ is a square. What is the degree measure of $\angle AFE$?



$(\mathbf{A})\ 160 \qquad (\mathbf{B})\ 164 \qquad (\mathbf{C})\ 166 \qquad (\mathbf{D})\ 170 \qquad (\mathbf{E})\ 174$

**Problem 7**

A school has $100$ students and $5$ teachers. In the first period, each student is taking one class, and each teacher is teaching one class. The enrollments in the classes are $50$, $20$, $20$, $5$, and $5$. Let $t$ be the average value obtained if a teacher is picked at random and the number of students in their class is noted. Let $s$ be the average value obtained if a student is picked at random and the number of students in their class, including that student, is noted. What is $t-s$?

$(\mathbf{A})\ -18.5 \qquad (\mathbf{B})\ -13.5 \qquad (\mathbf{C})\ 0 \qquad (\mathbf{D})\ 13.5 \qquad (\mathbf{E})\ 18.5$

**Problem 8**

Let $M$ be the least common multiple of all the integers $10$ through $30,$ inclusive. Let $N$ be the least common multiple of $M, 32, 33, 34, 35, 36, 37, 38, 39,$ and $40$. What is the value of $\frac{N}{M}$?

$(\mathbf{A})\ 1 \qquad (\mathbf{B})\ 2 \qquad (\mathbf{C})\ 37 \qquad (\mathbf{D})\ 74 \qquad (\mathbf{E})\ 2886$

**Problem 9**

A right rectangular prism whose surface area and volume are numerically equal has edge lengths $\log_{2}x$, $\log_{3}x$, and $\log_{4}x$. What is $x$?

$(\mathbf{A})\ 2\sqrt{6} \qquad (\mathbf{B})\ 6\sqrt{6} \qquad (\mathbf{C})\ 24 \qquad (\mathbf{D})\ 48 \qquad (\mathbf{E})\ 576$

**Problem 10**

The base-nine representation of the number $N$ is $27{,}006{,}000{,}052_{\text{nine}}$. What is the remainder when $N$ is divided by $5$?

$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ 1 \qquad \textbf{(C)}\ 2 \qquad \textbf{(D)}\ 3 \qquad \textbf{(E)}\ 4$

**Problem 11**

Consider two concentric circles of radius $17$ and radius $19$. The larger circle has a chord, half of which lies inside the smaller circle. What is the length of the chord in the larger circle?

$\textbf{(A)}\ 12\sqrt{2} \qquad \textbf{(B)}\ 10\sqrt{3} \qquad \textbf{(C)}\ \sqrt{17 \cdot 19} \qquad \textbf{(D)}\ 18 \qquad \textbf{(E)}\ 8\sqrt{6}$

**Problem 12**

What is the number of terms with rational coefficients among the $1001$ terms in the expansion of $\left(x\sqrt[3]{2}+y\sqrt{3}\right)^{1000}$?

$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ 166 \qquad \textbf{(C)}\ 167 \qquad \textbf{(D)}\ 500 \qquad \textbf{(E)}\ 501$

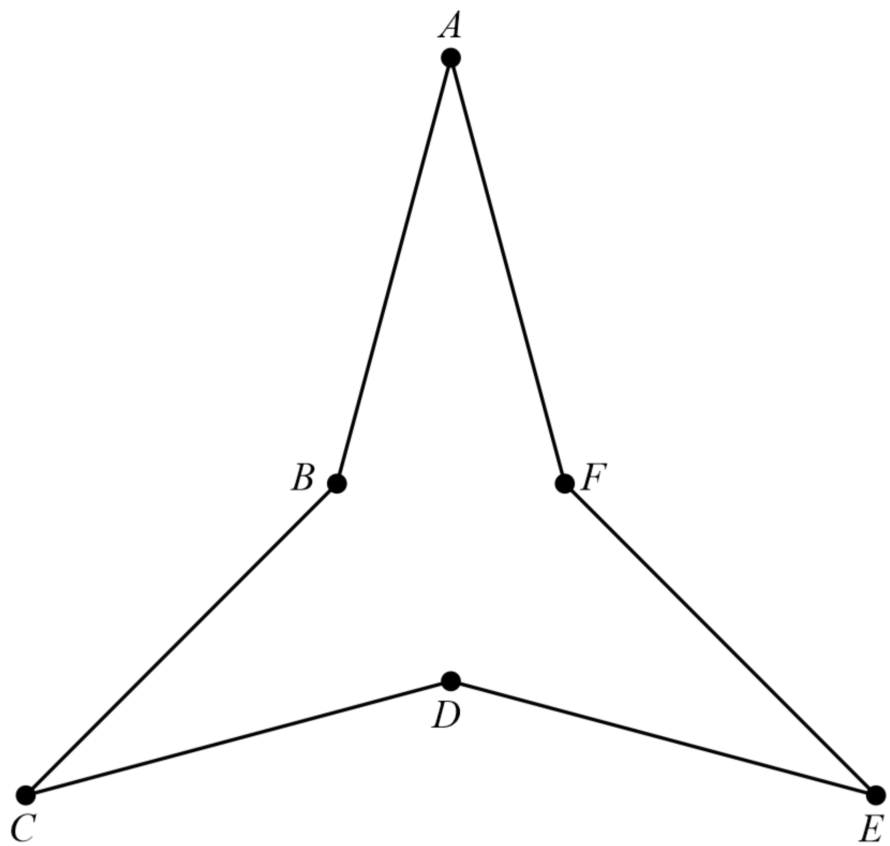
**Problem 13**

The angle bisector of the acute angle formed at the origin by the graphs of the lines $y = x$ and $y = 3x$ has equation $y = kx$. What is $k$?

$\textbf{(A)}\ \frac{1+\sqrt{5}}{2} \qquad \textbf{(B)}\ \frac{1+\sqrt{7}}{2} \qquad \textbf{(C)}\ \frac{2+\sqrt{3}}{2} \qquad \textbf{(D)}\ 2 \qquad \textbf{(E)}\ \frac{2+\sqrt{5}}{2}$

**Problem 14**

In the figure, equilateral hexagon $ABCDEF$ has three nonadjacent acute interior angles that each measure $30^\circ$. The enclosed area of the hexagon is $6\sqrt{3}$. What is the perimeter of the hexagon?




$(\textbf{A})\ 4 \qquad (\textbf{B})\ 4\sqrt{3} \qquad (\textbf{C})\ 12 \qquad (\textbf{D})\ 18 \qquad (\textbf{E})\ 12\sqrt{3}$

**Problem 15**

Recall that the conjugate of the complex number $w = a + bi$, where $a$ and $b$ are real numbers and $i = \sqrt{-1}$, is the complex number $\overline{w} = a - bi$. For any complex number $z$, let $f(z) = 4i\overline{z}$. The polynomial

$$P(z) = z^4 + 4z^3 + 3z^2 + 2z + 1$$

has four complex roots: $z_1$, $z_2$, $z_3$, and $z_4$. Let

$$Q(z) = z^4 + Az^3 + Bz^2 + Cz + D$$

be the polynomial whose roots are $f(z_1)$, $f(z_2)$, $f(z_3)$, and $f(z_4)$, where the coefficients $A$, $B$, $C$, and $D$ are complex numbers. What is $B + D$?

$\textbf{(A)}\ -304 \qquad \textbf{(B)}\ -208 \qquad \textbf{(C)}\ 12i \qquad \textbf{(D)}\ 208 \qquad \textbf{(E)}\ 304$

**Problem 16**

An organization has $30$ employees, $20$ of whom have a brand A computer while the other $10$ have a brand B computer. For security, the computers can only be connected to each other and only by cables. The cables can only connect a brand A computer to a brand B computer. Employees can communicate with each other if their computers are directly connected by a cable or by relaying messages through a series of connected computers. Initially, no computer is connected to any other. A technician arbitrarily selects one computer of each brand and installs a cable between them, provided there is not already a cable between that pair. The technician stops once every employee can communicate with every other. What is the maximum possible number of cables used?

$\textbf{(A)}\ 190 \qquad \textbf{(B)}\ 191 \qquad \textbf{(C)}\ 192 \qquad \textbf{(D)}\ 195 \qquad \textbf{(E)}\ 196$

**Problem 17**

For how many ordered pairs $(b, c)$ of positive integers does neither $x^2 + bx + c = 0$ nor $x^2 + cx + b = 0$ have two distinct real solutions?

$\textbf{(A)}\ 4 \qquad \textbf{(B)}\ 6 \qquad \textbf{(C)}\ 8 \qquad \textbf{(D)}\ 12 \qquad \textbf{(E)}\ 16$

**Problem 18**

Each of $20$ balls is tossed independently and at random into one of $5$ bins. Let $p$ be the probability that some bin ends up with $3$ balls, another with $5$ balls, and the other three with $4$ balls each. Let $q$ be the probability that every bin ends up with $4$ balls. What is $\frac{p}{q}$?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 4 \qquad \textbf{(C)}\ 8 \qquad \textbf{(D)}\ 12 \qquad \textbf{(E)}\ 16$

**Problem 19**

Let $x$ be the least real number greater than $1$ such that

$$\sin x = \sin \left(x^2\right),$$

where the arguments are in degrees. What is $x$ rounded up to the closest integer?

$\textbf{(A)}\ 10 \qquad \textbf{(B)}\ 13 \qquad \textbf{(C)}\ 14 \qquad \textbf{(D)}\ 19 \qquad \textbf{(E)}\ 20$

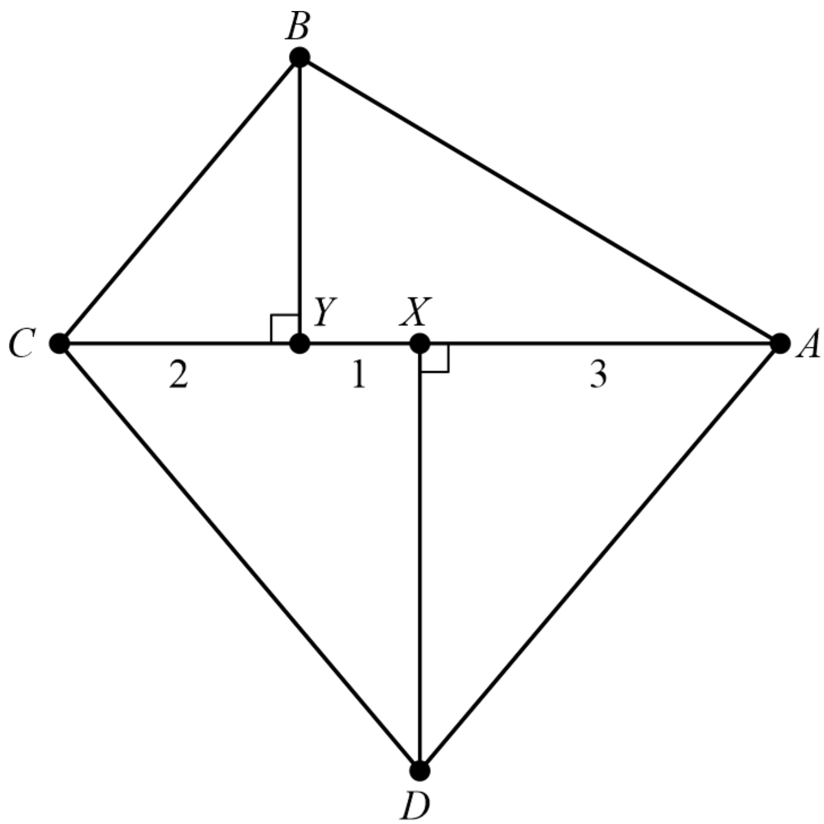
**Problem 20**

For each positive integer $n$, let $f_1(n)$ be twice the number of positive integer divisors of $n$, and for $j \ge 2$, let $f_j(n) = f_1(f_{j-1}(n))$. For how many values of $n \le 50$ is $f_{50}(n) = 12$?

$\textbf{(A)}\ 7 \qquad \textbf{(B)}\ 8 \qquad \textbf{(C)}\ 9 \qquad \textbf{(D)}\ 10 \qquad \textbf{(E)}\ 11$

## Problem 21

Let $ABCD$ be an isosceles trapezoid with $\overline{BC} \parallel \overline{AD}$ and $AB = CD$. Points $X$ and $Y$ lie on diagonal $\overline{AC}$ with $X$ between $A$ and $Y$ , as shown in the figure. Suppose $\angle AXD = \angle BYC = 90^\circ$ , $AX = 3$, $XY = 1$, and $YC = 2$. What is the area of $ABCD$?




$\textbf{(A)}\ 15 \qquad \textbf{(B)}\ 5\sqrt{11} \qquad \textbf{(C)}\ 3\sqrt{35} \qquad \textbf{(D)}\ 18 \qquad \textbf{(E)}\ 7\sqrt{7}$

## Problem 22

Azar and Carl play a game of tic-tac-toe. Azar places an $\textsf{X}$ in one of the boxes in a $3$-by-$3$ array of boxes, then Carl places an $\textsf{O}$ in one of the remaining boxes. After that, Azar places an $\textsf{X}$ in one of the remaining boxes, and so on until all $9$ boxes are filled or one of the players has $3$ of their symbols in a row---horizontal, vertical, or diagonal---whichever comes first, in which case that player wins the game. Suppose the players make their moves at random, rather than trying to follow a rational strategy, and that Carl wins the game when he places his third O. How many ways can the board look after the game is over?

$\textbf{(A) } 36 \qquad \textbf{(B) } 112 \qquad \textbf{(C) } 120 \qquad \textbf{(D) } 148 \qquad \textbf{(E) } 160$

**Problem 23**

A quadratic polynomial $p(x)$ with real coefficients and leading coefficient $1$ is called *disrespectful* if the equation $p(p(x)) = 0$ is satisfied by exactly three real numbers. Among all the disrespectful quadratic polynomials, there is a unique such polynomial $\tilde{p}(x)$ for which the sum of the roots is maximized. What is $\tilde{p}(1)$?

$\textbf{(A) } \frac{5}{16} \qquad \textbf{(B) } \frac{1}{2} \qquad \textbf{(C) } \frac{5}{8} \qquad \textbf{(D) } 1 \qquad \textbf{(E) } \frac{9}{8}$

**Problem 24**

Convex quadrilateral $ABCD$ has $AB = 18$, $\angle A = 60^\circ$, and $\overline{AB} \parallel \overline{CD}$. In some order, the lengths of the four sides form an arithmetic progression, and side $\overline{AB}$ is a side of maximum length. The length of another side is $a$. What is the sum of all possible values of $a$?

$\textbf{(A) } 24 \qquad \textbf{(B) } 42 \qquad \textbf{(C) } 60 \qquad \textbf{(D) } 66 \qquad \textbf{(E) } 84$

**Problem 25**

Let $m \geq 5$ be an odd integer, and let $D(m)$ denote the number of quadruples $(a_1, a_2, a_3, a_4)$ of distinct integers with $1 \leq a_i \leq m$ for all $i$ such that $m$ divides $a_1 + a_2 + a_3 + a_4$. There is a polynomial

$$q(x) = c_3x^3 + c_2x^2 + c_1x + c_0$$

such that $D(m) = q(m)$ for all odd integers $m \geq 5$. What is $c_1$?

$\textbf{(A)}\ -6 \qquad \textbf{(B)}\ -1 \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 6 \qquad \textbf{(E)}\ 11$